

EXAM I, MTH 320, Fall 2016

Ayman Badawi

QUESTION 1. (i) We know that $(Z, +)$ is cyclic. Prove that $F = (Z, +) \times (Z, +)$ is not a cyclic (Some of you have the right idea but ...)

Proof. Deny. Then $F =< (a, b) >$ for some $a, b \in Z$. It is clear that $a \neq 0$, and $b \neq 0$. Since $(1, 0) \in F$, there must exist $k \in Z$ such that $(1, 0) = (a, b)^k = (ak, bk)$. Hence $bk = 0$ and $ak = 1$. Since $bk = 0$ and $b \neq 0$, we conclude $k = 0$. But $(a, b)^0 = (0, 0) \neq (1, 0)$. A contradiction. Thus F is not cyclic.

(ii) Give me an example of an abelian group with 16 elements, say D, such that D has a subgroup H with exactly 8 elements, but D has no elements of order 8.

Solution: Let $D = (Z_4, +) \times (Z_4, +)$. We know that $|(a, b)| = LCM[|a|, |b|]$. Hence each element in D is of order 1, 2, or 4. Now $H = \{0, 2\}$ is a subgroup of Z_4. Thus $Z_4 \times H$ is a subgroup of D with 8 elements.

(iii) Let D be an abelian group such that D has a subgroup H with 10 elements. Given that D has an element a of order 2 where $a \notin H$. Prove that D has a subgroup of order 20.

Proof. Let $F = H \cup a * H$. We know $H \cap a * H = \emptyset$ and $|F| = 20$. Hence we show that F is closed. Let $x, y \in F$. Then $x = a^i * h_1, y = a^k * h_2$ where $0 \leq i, k \leq 2, h_1, h_2 \in H$. Thus $x * y = a^{i+k(mod2)}h_1h_2 \in F$.

(iv) We know that if a, b are elements of a group $(D, *)$ such that $a * b = b * a$ and $gcd(|a|, |b|) = 1$, then $|a * b| = |a||b|$. Give me an example of a group D that has two elements, say a, b, such that $gcd(|a|, |b|) = 1$ but $|a * b| \neq |a||b|$.

Solution: Let $a = (1\ 2\ 3), b = (2\ 3) \in S_3$. Then $|a| = 3$ and $|b| = 2$. $aob = (1\ 2)$. Thus $|aob| = 2$, where $|a||b| = 6$

(v) Let $(D, *)$ be a group and $a, b \in D$ such that $a * b = b * a$. Prove that $a^{-1} * b^{-1} = b^{-1} * a^{-1}$.

Proof. Since $a*b = b*a$, we have $(a*b)^{-1} = (b*a)^{-1}$. We know that $(a*b)^{-1} = b^{-1}*a^{-1}$ and $(b*a)^{-1} = a^{-1}*b^{-1}$. Thus $a^{-1} * b^{-1} = b^{-1} * a^{-1}$.

(vi) Let $(D, *)$ be a group such that $a^2 = e$ for every $a \in D$. Prove that D is an abelian group.

Proof. Since $a^2 = e$ for every $a \in D$, we conclude that $a = a^{-1}$ for every $a \in D$. Now let $x, y \in D$. Since $x * y \in D$, we have $(x * y)^2 = (x * y) * (x * y) = e$. Thus $x * y = y^{-1} * x^{-1} = y * x$ (since $y^{-1} = y$ and $x^{-1} = x$

(vii) ((All of you - 2) got it right just straightforward class notes, see your notes)

Is $U(10) \times (Z_7, +)$ cyclic? Explain briefly.

b. Is $U(15) \times (Z_9, +)$ cyclic? Explain briefly.

c. Let $F = (Z_{12}, +)$ and $H = \{0, 3, 6, 9\}$. Find all left cosets of H

d. Let $V = (1\ 3\ 4)o(2\ 5\ 6)$ Find $|v|$

e. Let $V = (1\ 3\ 5)o(2\ 3\ 4\ 5)$. Find $|v|$.

ℍ **Faculty information**

Ayman Badawi, Department of Mathematics & Statistics, American University of Sharjah, P.O. Box 26666, Sharjah, United Arab Emirates.
E-mail: abadawi@aus.edu, www.ayman-badawi.com